STRICTLY CONFIDENTIAL

System Reliability and Free Riding

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定義

- Total effort (ex. 城門的強度)

  $F(x_1, x_2) = x_1 + x_2$
- Weakest link (ex. 城牆的高度)

  $F(x_1, x_2) = \min(x_1, x_2)$
- Best shot (ex. 城牆裡還有城牆)

  $F(x_1, x_2) = \max(x_1, x_2)$
- The probability of successful operation of the system

  $P(F(x_1, x_2))$
- The expected payoff to agent *i*

  $P(F(x_1, x_2))v_i - c_i x_i$
- The social payoff

  $P(F(x_1, x_2))[v_1 + v_2] - c_1 x_1 - c_2 x_2$

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Note

P(F)機率函數有三大特點

(1) 可微→連續

(2) 在F的範圍內遞增

(3) Concave

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Nash equilibria (Total effort)

- Agent 1 choose x_1 to solve

  $\max_{x_1} v_1 P(x_1 + x_2) - c_1 x_1$
- First-order conditions

  $v_1 P'(x_1 + x_2) = c_1$
- Let *G* be the inverse of the derivative of *P*

  $x_1 + x_2 = G(c_1 / v_1)$
- Defining

  $\bar{x}_1 = G(c_1 / v_1)$
- Reaction functions

$x_1 = f_1(x_2) = \bar{x}_1 - x_2 \;\; but \; if \;\; x_2 > \bar{x}_1 \;\; then \; we \; have \; x_1 = 0$

$x_2 = f_2(x_1) = \bar{x}_2 - x_1 \;\; but \; if \;\; x_1 > \bar{x}_2 \;\; then \; we \; have \; x_2 = 0$

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Nash equilibria (Total effort)

- Reaction functions (suppose $\bar{x}_1 > \bar{x}_2$)

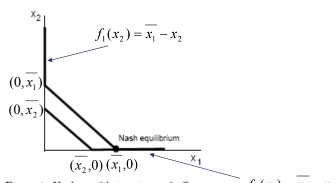

Figure 1: Nash equilibrium in total effort case.

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Note

P(F)機率函數有三大特點

(1) 可微→連續

(2) 在F的範圍內遞增

(3) Concave

因此 $y = P'(x)$ 爲遞減函數

推得 $G(y) = x$ 也同爲遞減函數

所以我們有下列推論

$\bar{x}_1 = G\left(\frac{c_1}{v_1}\right), \quad \bar{x}_2 = G\left(\frac{c_2}{v_2}\right)$

$if \; \frac{v_2}{c_2} > \boxed{\frac{v_1}{c_1}}$ ← 我們稱之爲benefit/cost ratio

$then \; \frac{c_1}{v_1} > \frac{c_2}{v_2} \Rightarrow \bar{x}_2 > \bar{x}_1$

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Fact 1

In the case of total effort, system reliability is determined by the agent with the highest benefit-cost ratio.

All other agents free ride on this agent.

Nash equilibria (Weakest link)

- **Agent 1 choose x_1 to solve**

  $\max_{x_1} v_1 P(\min(x_1, x_2)) - c_1 x_1$

- **if** $x_2 < \overline{x_1}$ 　$v_1 P(x_2) - c_1 x_2 > v_1 P(x_2) - c_1(x_2 + d)$

  假如x_1選得比x_2 大→不會比設 $x_1 = x_2$ **好**

  因此只考慮 $0 \le x_1 \le x_2$ **,怎樣選x_1爲最好?**

  suppose $x_1 < x_2$

  因爲 $v_1 P(x_1) - c_1 x_1 < v_1 P(x_2) - c_1 x_2$

  觀察出來設 $x_1 = x_2$ **爲最好**

- **Otherwise (** $x_2 \ge \overline{x_1}$ **)**

  設 $x_1 = \overline{x_1}$

排除x_1>x_2後,與該問題等價 →

$$\max_{x_1} v_1 P(x_1) - c_1 x_1$$
$$s.t.$$
$$0 \le x_1 \le x_2$$
$$\overline{x_1} \le x_2$$

Why?

Nash equilibria (Weakest link)

- **Reaction functions (suppose** $\overline{x_2} > \overline{x_1}$ **)**

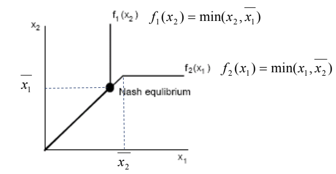

Figure 2: Nash equilibrium in weakest link case.

Fact 2

In the weakest-link case, system reliability is determined by the agent with the lowest benefit-cost ratio.

Nash equilibria (Best shot)

- **Agent 1 choose x_1 to solve (Suppose** $\overline{x_2} > \overline{x_1}$ **)**

  $\max_{x_1} v_1 P(\max(x_1, x_2)) - c_1 x_1$

- $x_1 = 0, x_2 = \overline{x_2}$ **is a Nash equilibrium**

  因爲假如 $x_1 = 0$ **則** $\max_{x_2} v_2 P(\max(0, x_2)) - c_2 x_2 = \max_{x_2} v_2 P(x_2) - c_2 x_2$

  倘若假如 $x_2 = \overline{x_2}$ **則因爲下列關係,會使得** $x_1 = 0$

  $v_1 P(\overline{x_2}) > v_1 P(\overline{x_2}) - c_1 \overline{x_2} > v_1 P(\overline{x_2} + d) - c_1(\overline{x_2} + d)$

- **Sometimes** $x_1 = \overline{x_1}, x_2 = 0$ **也是Nash equilibrium**

  因爲假如 $x_2 = 0$ **則** $\max_{x_1} v_1 P(\max(x_1, 0)) - c_1 x_1 = \max_{x_1} v_1 P(x_1) - c_1 x_1$

  可是是否 $x_1 = \overline{x_1}$ **時, x_2 也會選對應的 0 ?**

  (須符合特定條件!!)

  $if\ v_2 P(\overline{x_1}) > v_2 P(\overline{x_2}) - c_2 \overline{x_2}$

Nash equilibria (Best shot)

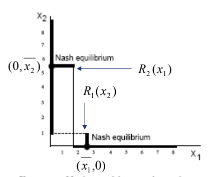

Figure 3: Nash equilibria in best-shot case.

Social optimum (Total effort)

- **The social problem solves**
  $\max_{x_1 x_2} P(x_1+x_2)[v_1+v_2]-c_1x_1-c_2x_2$
- **First-order conditions**
  $P'(x_1+x_2)[v_1+v_2]\le c_1$
  $P'(x_1+x_2)[v_1+v_2]\le c_2$
- **Let C_{min} = {C_1,C_2}**
  $x_1^*+x_2^*=G(c_{\min}/(v_1+v_2))$

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Fact 3

In the total effort case, there is always too little effort exerted in the Nash equilibrium as compared with the optimum. Furthermore, when $v_2/c_2 > v_1/c_1$ but $c_1<c_2$, the "wrong" agent exerts the effort.

Note: $G(y)=x$ **爲遞減函數**

Social $x_1^*+x_2^*=G(c_{\min}/(v_1+v_2))$ **V.S.**

Private $x_1^*+x_2^*=G(c_2/v_2)+0 \quad if\ v_2/c_2>v_1/c_1$

when $v_2/c_2 > v_1/c_1$ but $c_1<c_2$

Social $x_1^*+x_2^*=G(c_1/(v_1+v_2))$

Private $x_1^*+x_2^*=G(c_2/v_2)+0$

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Social optimum (Best Shot)

- **The social problem solves**
  $\max_{x_1 x_2} P(\max(x_1,x_2))[v_1+v_2]-c_1x_1-c_2x_2$
- **不可能設** $x_1=x_2=k\ if\ k\ne 0$
  $P(k)[v_1+v_2]-c_1k-c_2k\le P(k)[v_1+v_2]-c_1k$
  $P(k)[v_1+v_2]-c_1k-c_2k\le P(k)[v_1+v_2]-c_2k$
- **換言之,Either** $x_1=something\ (x_2=0)$ **or** $x_2=something\ (x_1=0)$
  $\max_{x_1 x_2} P(\max(x_1,x_2))[v_1+v_2]-c_1x_1-c_2x_2$
  與下列問題等價
  $let\ f_1=\max_{x_1} P(x_1)[v_1+v_2]-c_1x_1,\ f_2=\max_{x_2} P(x_2)[v_1+v_2]-c_2x_2$

  試求 $\max\{f_1,f_2\}$ **??**
  The answer is $x_1^*+x_2^*=G(c_{\min}/(v_1+v_2))$

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補充

- **Suppose** $c_{\min}=c_1$
  $P(x_1^*)[v_1+v_2]-c_1x_1^*>$
  $P(x_2^*)[v_1+v_2]-c_1x_2^*>$
  $P(x_2^*)[v_1+v_2]-c_1x_2^*-\delta x_2^*=P(x_2^*)[v_1+v_2]-c_2x_2^*$
  $where\ c_2=c_1+\delta$
- **Hence, choose** $x_1^*=G(c_{\min}/(v_1+v_2))$ **and** $x_2^*=0$

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Social optimum (Weakest link)

- **The social problem solves**
  $\max_{x_1 x_2} P(\min(x_1,x_2))[v_1+v_2]-c_1x_1-c_2x_2$
- **不可能設** $x_1\ne x_2$
  $suppose\ x_2=x_1+d>x_1$
  $P(x_1)[v_1+v_2]-c_1x_1-c_2(x_1+d)<P(x_1)[v_1+v_2]-c_1x_1-c_2x_1$
- **換言之,** $x_1=x_2$
  $\max_{x_1 x_2} P(\min(x_1,x_2))[v_1+v_2]-c_1x_1-c_2x_2$
  與下列問題等價
  $\max_{x} P(x)[v_1+v_2]-[c_1+c_2]x$
- **First-order conditions**
  $P'(x)[v_1+v_2]=[c_1+c_2]$
  $x_1^*=x_2^*=x=G((c_1+c_2)/(v_1+v_2))$

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Fact 4

The probability of success in the socially optimal solution is always lower in the case of weakest link than in the case of total effort.

Note: $G(y)=x$ **爲遞減函數**

Total effort $P(x_1^*+x_2^*)=P(G(c_{\min}/(v_1+v_2)))$ **V.S.**

Weakest link $P(\min(x_1^*,x_2^*))=P(G((c_1+c_2)/(v_1+v_2)))$

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Identical Values, Different Costs

- Let n be the number of agents, set $v_i = 1$ for all $i = 1, \ldots, n$
- In the total-effort case, the social problem solves
  $$\max P(\sum_{i=1}^{n} x_i)\left[\sum_{i=1}^{n} v_i\right] - \sum_{i=1}^{n} c_i x_i$$
- First-order conditions (define $x = x_1+x_2+\ldots+x_n$)
  $$nP'(x) \le c_i \ \ for\ i = 1, \cdots, n$$
- 換言之,social optimal is determined by
  $$nP'(x) = \min c_i$$
- Private optimum is determined by
  $$P'(x) = \min c_i \quad (in\ fact, x = x_i\ and\ the\ others\ are\ zeros)$$

Note that in the case of total effort, system reliability is determined by the agent with the highest benefit-cost ratio. (Fact 1)

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Identical Values, Different Costs

- In the weakest-link case, the social problem
  $$\max P(\min\{x_1, x_2, \cdots, x_n\})\left[\sum_{i=1}^{n} v_i\right] - \sum_{i=1}^{n} c_i x_i$$
- Obviously, $x_1 = x_2 = \cdots = x_n$ 爲optimal choice

  換言之,該問題可轉成
  $$\max_x nP(x) - x\sum_{i=1}^{n} c_i$$
- Hence, the social optimal is determined by
  $$nP'(x) = \sum_{i=1}^{n} c_i \quad or \quad P'(x) = \frac{1}{n}\sum_{i=1}^{n} c_i = \bar{c}$$
- The private optimal is determined by
  $$P'(x) = \max c_i$$

Note that in the weakest-link case, system reliability is determined by the agent with the lowest benefit-cost ratio.

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Fact 5

Systems will become increasingly reliable as the number of agents increases in the total effort case, but increasingly unreliable as the number of agents increases in the weakest link case.

Note: $G(y) = x$ 爲遞減函數

Total effort $P'(x) = \min c_i \Rightarrow x = G(\min c_i)$ **V.S.**

Weakest link $P'(x) = \max c_i \Rightarrow x = G(\max c_i)$

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Increasing the number of agents

- Suppose $v_i = c_i = 1$ for all $i = 1, \ldots, n$
- In the total-effort case, the social optimal is determined by
  $$nP'(\sum_{i=1}^{n} x_i) = 1 \quad or \quad \sum_{i=1}^{n} x_i = G(1/n)$$
- Nash equilibrium satisfies (Private optimum)
  $$P'(\sum_{i=1}^{n} x_i) = 1 \quad or \quad \sum_{i=1}^{n} x_i = G(1)$$

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Fact 6

In the total efforts case with identical agents, the Nash outcome remains constant as the number of agents is increased, but the socially optimal amount of effort increases.

Note: $G(y) = x$ 爲遞減函數

Social optimal $\sum_{i=1}^{n} x_i = G(1/n)$

V.S.

Nash equilibrium $\sum_{i=1}^{n} x_i = G(1)$

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Increasing the number of agents

- Suppose $v_i = c_i = 1$ for all $i = 1, \ldots, n$
- In the weakest-link case, the social optimal is determined by $nP'(x) = n \quad or \quad x = G(1)$
- Nash equilibrium satisfies (Private optimum)
  $$P'(x) = 1 \quad or \quad x = G(1)$$

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Fact 7

In the weakest-link case with identical agents, the socially optimal reliability and the Nash reliability are identical, regardless of the number of agents.

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Fines and liability (Total effort)

- **If we impose a cost of v_2 on agent 1 in the event that the system fails, then agent 1 will want to maximize**

  $$v_1 P(x_1 + x_2) - v_2[1 - P(x_1 + x_2)] - c_1 x_1$$
- **The first order condition is**

  $$(v_1 + v_2)P'(x_1 + x_2) = c_1$$
- **This is the condition for social optimality (why?)**

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Notes

- **Recall that the social optimality is determined by**

  $$x_1^* + x_2^* = G(c_{\min}/(v_1 + v_2))$$
- **From the first order condition for the case of fines for agent *i***

  $$(v_1 + v_2)P'(x_1 + x_2) = c_i, \quad where\ i = 1,2$$
- **Then we have two response functions as follows:**

  $$x_1 + x_2 = G\left(\frac{c_1}{(v_1 + v_2)}\right) \Rightarrow x_1 = R_2(x_2) = G\left(\frac{c_1}{(v_1 + v_2)}\right) - x_2$$

  $$x_1 + x_2 = G\left(\frac{c_2}{(v_1 + v_2)}\right) \Rightarrow x_2 = R_1(x_2) = G\left(\frac{c_2}{(v_1 + v_2)}\right) - x_1$$
- **By Fact 1, this implies the social optimality**

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Fact 8

A fine equals to the costs imposed on the other agents should be imposed on the agent who has the lowest cost of reducing the probability of failure.

P.S:

$$\left(\sum_{i=1}^{n} v_i\right)P'\left(\sum_{i=1}^{n} x_i\right) = c_k, \quad where\ c_k = \min\{c_1, c_2, \cdots, c_n\}$$

$$(v_j)P'\left(\sum_{i=1}^{n} x_i\right) = c_j, \quad where\ j = 1,2,\cdots,n\ but\ j \neq k$$

the above equations imply the following result

$$\frac{v_1 + v_2 + \cdots + v_n}{c_{\min}} > \frac{v_j}{c_j}$$

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Fines and liability (Total effort)

- **We could consider a strict liability rule**
- **The amount charged in the case of system failure is paid to the other agent**
- **Suppose "fine" is paid to agent 2, his optimization problem becomes**

  $$v_2 P(x_1 + x_2) + v_2[1 - P(x_1 + x_2)] - c_2 x_2 \Rightarrow v_2 - c_2 x_2$$
- **So agent 2 will want to set**

  $$x_2 = 0$$
- **But this is true in the social optimum as well**

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Fines and liability (Weakest link)

- **One way to do this is to make each agent face the other's marginal cost, in addition to facing a fine in case of system failure.**

  $$v_1 P(x) - [1 - P(x)]v_2 - c_1 x_1 - c_2 x_1 \ \ for\ agent\ 1$$

  $$v_2 P(x) - [1 - P(x)]v_1 - c_1 x_2 - c_2 x_2 \ \ for\ agent\ 2$$

  $$where\ x = \min\{x_1, x_2\}$$
- **Agent 1 and agent 2 want to choose $x = x_1 = x_2$ determined by**

  $$(v_1 + v_2)P'(x) = c_1 + c_2$$
- **This is the condition for social optimality**

  $$x_1^* = x_2^* = x = G((c_1 + c_2)/(v_1 + v_2))$$

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Fines and liability (Weakest link)

- ***Each*** **must compensate the other in the case of system failure**

$$\max_{x_1} v_1 P(x) - [1 - P(x)]v_2 + [1 - P(x)]v_1 - c_1 x_1$$
$$\max_{x_2} v_2 P(x) - [1 - P(x)]v_1 + [1 - P(x)]v_2 - c_2 x_2$$
$$\textit{where } x = \min\{x_1, x_2\}$$

- ***Simplifying***

$$\max_{x_1} v_1 - v_2 + v_2 P(x) - c_1 x_1$$
$$\max_{x_2} v_2 - v_1 + v_1 P(x) - c_2 x_2$$

- **Suppose $v_1 = v_2$ and $c_1 = c_2$** *(Symmetric case)*

$$v_2 P'(x) = c_1 \quad \textit{and} \quad v_1 P'(x) = c_2$$

- **This is the condition for social optimality**

$$x_1^* = x_2^* = x = G((c_1 + c_2)/(v_1 + v_2))$$

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Fines and liability (Weakest link)

- **If we are *not* in the symmetric case, the equilibrium will be determined by**

$$\max\left\{\frac{c_1}{v_2}, \frac{c_2}{v_1}\right\} \quad \textit{for} \quad v_2 P'(x) = c_1 \quad \textit{and} \quad v_1 P'(x) = c_2$$

- **In this case, strict liability *does not* result in the social optimum**
- **The resolution is to use the *negligence rule***

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Fines and liability (Weakest link)

- **Let x^* be the socially optimal effort level**

$$\max_{x_1 x_2} P(x_1 + x_2)[v_1 + v_2] - c_1 x_1 - c_2 x_2$$

- **If the due care standard is set at $\bar{x} = x^*$ then**

$x_1 = x_2 = \bar{x}$ **is a Nash equilibrium**

- **To prove this, assume that $x_2 = \bar{x}$**
- **We will never have $x_1 > \bar{x}$**

(1) $\min(x_1, x_2)$ **no impact on the probability of system failure**

(2) $c_1 x_1$ **incurs positive cost**

- **Will agent 1 ever to choose $x_1 < \bar{x}$?**

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Fines and liability (Weakest link)

agent 1's objective function

$$v_1 P(x_1) - (1 - P(x_1))v_2 - c_1 x_1$$

Computing the derivative

It's optimal when $(v_1 + v_2)P'(x_1) - c_1 = 0$

But we have $(v_1 + v_2)P'(x^*) - c_1 = c_2$

Because the concavity of $P(x)$ *and* $x_1 < \bar{x} = x^*$

This means $(v_1 + v_2)P'(x_1) - c_1 > (v_1 + v_2)P'(x^*) - c_1 = c_2$

Hence agent1 will want to increase his level of effort

when $x_1 < \bar{x}$

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Fact 9

In the case of weakest link, strict liability is not adequate in general to achieve the socially optimal level of effort, and one must use a negligence rule to induce the optimal effort.

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