

Lecture Note II-3 Static Games of Incomplete Information

- Static Bayesian Game
- Bayesian Nash Equilibrium
- Applications: Auctions
- The Revelation Principle

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Games of incomplete information

- Also called **Bayesian Games**
- At least one player is **uncertain** about another player's **payoff function**
- Example of static game of incomplete information: A sealed-bid auction
 - Each bidder knows his or her own valuation for the good being sold but does not know any other bidder's valuation
 - Bids are submitted in sealed envelopes (players' moves can be thought of as simultaneous)

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Example: Cournot Competition under Asymmetric Information

- Cournot duopoly model with inverse demand given by $P(Q)=a-Q$, where $Q=q_1+q_2$
- Firm 1's cost function is $C_1(q_1)=cq_1$
- Firm 2's cost function is $C_2(q_2)=c_Hq_2$ with probability θ and $C_2(q_2)=c_Lq_2$ with probability $1-\theta$
- **Information is asymmetric**: firm 2 knows its cost function and firm 1's, but firm 1 knows its cost function and only that firm 2's marginal cost is c_H with probability θ and c_L with probability $1-\theta$
- All of this is common knowledge: firm 1 knows that firm 2 has superior information, firm 2 knows that firm 1 knows this, and so on

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Cournot Competition under Asymmetric Information (cont')

- Let $q_2(c_H)$ and $q_2(c_L)$ denote firm 2's quantity choice as a function of its cost, and let q_1^* denote firm 1's single quantity choice
- If firm 2's cost is high, it will choose $q_2(c_H)$ to solve

$$\max_{q_2} [(a - q_1^* - q_2) - c_H] q_2$$

- If firm 2's cost is low, it will choose $q_2(c_L)$ to solve

$$\max_{q_2} [(a - q_1^* - q_2) - c_L] q_2$$

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Cournot Competition under Asymmetric Information (cont')

- Firm 1 chooses q_1^* to solve

$$\max_{q_1} \theta [(a - q_1 - q_2^*(c_H)) - c] q_1 + (1 - \theta) [(a - q_1 - q_2^*(c_L)) - c] q_1$$

- The first-order conditions for these three optimization problem are

$$q_2^*(c_H) = \frac{a - q_1^* - c_H}{2}$$

$$q_2^*(c_L) = \frac{a - q_1^* - c_L}{2}$$

$$q_1^* = \frac{\theta [(a - q_2^*(c_H)) - c] + (1 - \theta) [(a - q_2^*(c_L)) - c]}{2}$$

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Cournot Competition under Asymmetric Information (cont')

- The solutions to the three first-order conditions are

$$q_2^*(c_H) = \frac{a - 2c_H + c}{3} + \frac{1 - \theta}{6} (c_H - c_L); \quad q_1^* = \frac{a - 2c + \theta c_H + (1 - \theta) c_L}{3}$$

$$q_2^*(c_L) = \frac{a - 2c_L + c}{3} - \frac{\theta}{6} (c_H - c_L)$$

- If information is **symmetric** (player 1 knows player 2's cost (c_H or c_L), say c_2)

$$q_2^{**}(c_2) = \frac{a - 2c_2 + c}{3} \quad q_1^{**}(c_2) = \frac{a - 2c + c_2}{3}$$

Implication \Rightarrow $q_2^*(c_H) > q_2^{**}(c_H); q_2^*(c_L) < q_2^{**}(c_L)$
 $q_1^{**}(c_L) < q_1^* < q_1^{**}(c_H)$

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Static Bayesian Games

- Definition:** The normal-form representation of an n -player static Bayesian game specifies the players' **action spaces** A_1, \dots, A_n , their **type spaces** T_1, \dots, T_n , their **beliefs** p_1, \dots, p_n , and their **payoff functions** u_1, \dots, u_n . Player i 's **type** t_i is privately known by player i , determines player i 's payoff function, $u_i(a_1, \dots, a_n; t_i)$, and t_i is a member of the set of possible types, T_i . Player i 's belief $p_i(t_{-i}|t_i)$ describes i 's uncertainty about the $n-1$ other players' possible types, t_{-i} , given i 's own type, t_i . We denote this game by $G = \{A_1, \dots, A_n; T_1, \dots, T_n; p_1, \dots, p_n; u_1, \dots, u_n\}$

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Static Bayesian Games (cont')

- Example: Cournot game
 - The firms actions are their quantity choices, q_1 and q_2
 - Firm 2 has two possible profit or payoff functions

$$\pi_2(q_1, q_2; c_L) = [(a - q_1 - q_2) - c_L]q_2$$

$$\pi_2(q_1, q_2; c_H) = [(a - q_1 - q_2) - c_H]q_2$$
 - Firm 1 has only one possible payoff function

$$\pi_1(q_1, q_2; c) = [(a - q_1 - q_2) - c]q_1$$
 - Firm 2's type space is $T_2 = \{c_L, c_H\}$ and that firm 1's type space is $T_1 = \{c\}$

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Static Bayesian Games (cont')

- When player i 's payoff depend not only on the actions (a_1, \dots, a_n) but also on all the types (t_1, \dots, t_n) , we write this payoff as $u_i(a_1, \dots, a_n; t_1, \dots, t_n)$
- Assume nature draws a type vector $t = (t_1, \dots, t_n)$ according to the **prior probability distribution** $p(t)$ player i 's belief probability $p_i(t_{-i}|t_i)$ can be computed using Bayes' rule

$$p_i(t_{-i} | t_i) = \frac{p(t_{-i}, t_i)}{p(t_i)} = \frac{p(t_{-i}, t_i)}{\sum_{t_{-i} \in T_{-i}} p(t_{-i}, t_i)}$$

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Bayesian Nash Equilibrium

- Definition:** In the static Bayesian game $G = \{A_1, \dots, A_n; T_1, \dots, T_n; p_1, \dots, p_n; u_1, \dots, u_n\}$, a strategy for player i is a function $s_i(t_i)$ where for each type t_i in T_i , $s_i(t_i)$ specifies the action from the feasible set A_i that type t_i would choose if drawn by nature
 - i.e. **A strategy is a function from types to actions**
- Definition:** In the static Bayesian game $G = \{A_1, \dots, A_n; T_1, \dots, T_n; p_1, \dots, p_n; u_1, \dots, u_n\}$, the strategies $s^* = (s_1^*, \dots, s_n^*)$ are a (pure strategy) **Bayesian Nash equilibrium** if for each player i and for each of i 's types t_i in T_i , $s_i^*(t_i)$ solves

$$\max_{a_i \in A_i} \sum_{t_{-i} \in T_{-i}} u_i(s_1^*(t_1), \dots, s_{i-1}^*(t_{i-1}), a_i, s_{i+1}^*(t_{i+1}), \dots, s_n^*(t_n); t) p_i(t_{-i} | t_i)$$

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Example1: Battle of Sexes Game

		Player 2	
		Opera	Fight
Player 1	Opera	$2+t_c, 1$	$0, 0$
	Fight	$0, 0$	$1, 2+t_p$

- Player 1's payoff if both attend Opera is $2+t_c$, where t_c is **privately known** by player 1
- Player 2's payoff if both attend Fight is $2+t_p$, where t_p is **privately known** by player 2
- t_c and t_p are independent draws from a **uniform distribution on $[0, x]$**

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Example1: Battle of Sexes Game (cont')

- Static Bayesian game $G = \{A_c, A_p; T_c, T_p; p_c, p_p; u_c, u_p\}$
- The action spaces are $A_c = A_p = \{\text{Opera}, \text{Fight}\}$
- The type spaces are $T_c = T_p = [0, x]$
- The beliefs are $p_c(t_p) = p_p(t_c) = 1/x$ for all t_c and t_p
- Strategies
 - Player 1 plays Opera if t_c exceeds a critical value c , and plays Fight otherwise
 - With probability $(x-c)/x$ to play Opera
 - Player 2 plays Fight if t_p exceeds a critical value p , and plays Opera otherwise
 - With probability $(x-p)/x$ to play Fight

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Example 1: Battle of Sexes Game (cont')

- Player 1 's optimal strategy is play Opera if

$$\frac{p}{x}(2+t_c)+[1-\frac{p}{x}]\cdot 0 \geq \frac{p}{x}\cdot 0+[1-\frac{p}{x}]\cdot 1$$

$$\Leftrightarrow \frac{p}{x}(2+t_c) \geq 1-\frac{p}{x} \Leftrightarrow t_c \geq \frac{x}{p}-3=c$$

- Player 2 's optimal strategy is play Fight if

$$[1-\frac{c}{x}]\cdot 0+\frac{c}{x}(2+t_p) \geq [1-\frac{c}{x}]\cdot 1+\frac{c}{x}\cdot 0$$

$$\Leftrightarrow \frac{c}{x}(2+t_p) \geq 1-\frac{c}{x} \Leftrightarrow t_p \geq \frac{x}{c}-3=p$$

	Opera	Fight
Opera	2+t _c , 1	0, 0
Fight	0, 0	1, 2+t _p

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Example 1: Battle of Sexes Game (cont')

$$\Leftrightarrow p=c; p^2+3p-x=0 \Leftrightarrow \frac{x-c}{x} = \frac{x-p}{x} = 1 - \frac{-3+\sqrt{9+4x}}{2x}$$

$$\text{When } x \text{ is small } \Leftrightarrow \lim_{x \rightarrow 0} \frac{x-c}{x} = \lim_{x \rightarrow 0} \frac{x-p}{x} = \lim_{x \rightarrow 0} (1 - \frac{-3+\sqrt{9+4x}}{2x}) = \frac{2}{3}$$

Implication: player 1 play Opera with probability 2/3
player 2 play Fight with probability 2/3

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Mixed strategy Nash Equilibrium VS. Pure –Strategy Bayesian Nash Equilibrium

- A mixed- strategy Nash equilibrium is a game of complete information can be (almost always) be interpreted as a pure-strategy Bayesian Nash equilibrium in a closely related game with a little bit of incomplete information
- Example: Battle of Sexes

	q	1-q
	Opera	Fight
r Opera	2, 1	0, 0
1-r Fight	0, 0	1, 2

Two pure Nash equilibrium (Opera, Opera) and (Fight, Fight)
A mixed strategy Nash Equilibrium
(r, 1-r)=(2/3, 1/3) and (q, 1-q)=(1/3, 2/3)

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Application 1: An auction

- First-price, sealed-bid auction
 - The bidders simultaneously submit their bids. The higher bidders wins the good and pays the price she bid
 - In case of a tie, the winner is determined by a flip of a coin
- Two bidders, labeled $i=1,2$
 - Bidder i has value v_i for the good. The two bidders' valuations are independently and uniformly distributed on $[0,1]$
 - If bidder i gets the good and plays the price p , then i 's payoff is $v-p$

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Application 1: An auction

- Bayesian game
 - $G=\{A_1, A_2; T_1, T_2; p_1, p_2; u_1, u_2\}$
 - Action spaces $A_i=[0, \infty]$
 - Type space is $T_i=[0, 1]$
 - Player i 's action is to submit a bid b_i and her type is her valuation v_i
 - Player i 's payoff function

$$u_i(b_1, b_2; v_1, v_2) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ (v_i - b_i)/2 & \text{if } b_i = b_j \\ 0 & \text{if } b_i < b_j \end{cases}$$

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Application 1: An auction (cont')

- Player 1's strategy $b_1(v_1)$ is a best response to player 2's strategy $b_2(v_2)$ and vice versa
- The pair of $(b_1(v_1), b_2(v_2))$ is a Bayesian Nash equilibrium if for each v_i in $[0, 1]$, $b_i(v_i)$ solves

$$\max_{b_i} (v_i - b_i) \text{Prob}\{b_i > b_j(v_j)\} + \frac{1}{2}(v_i - b_i) \text{prob}\{b_i = b_j(v_j)\}$$

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Application 1: An auction (cont')

- Linear equilibrium: $b_i(v_i) = a_i + c_i v_i$
 - e.g. $b_1(v_1) = a_1 + c_1 v_1$ and $b_2(v_2) = a_2 + c_2 v_2$
- $$\max_{b_i} u_i = (v_i - b_i) \text{Prob}\{b_i > b_j(v_j)\} = (v_i - b_i) \cdot \frac{b_i - a_j}{c_j}$$
- s.t. $a_j \leq b_i \leq a_j + c_j$

Note:

$$\text{Prob}\{b_i > b_j(v_j)\} = \text{Prob}\left\{v_j < \frac{b_i - a_j}{c_j}\right\} = \frac{b_i - a_j}{c_j}$$

$$\Rightarrow b_i(v_i) = \begin{cases} (v_i + a_j)/2 & \text{if } v_i \geq a_j \\ a_j & \text{if } v_i < a_j \end{cases}$$

$$\Rightarrow b_i(v_i) = v_i/2$$

$$a_i = a_j = 0; c_i = c_j = 1/2$$

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Application 1: An auction (cont')

- Symmetric equilibrium: suppose player j adopts the strategy $b(\cdot)$, and assume $b(\cdot)$ is strictly increasing and differential

$$\max_{b_i} u_i = (v_i - b_i) \text{Prob}\{b_i > b_j(v_j)\}$$

$$b^{-1}(b_j) = v_j \text{ if } b_j = b(v_j)$$

$$\text{Prob}\{b_i > b(v_j)\} = \text{Prob}\{b^{-1}(b_i) > v_j\} = b^{-1}(b_i) = v_j$$

First order condition

$$-b^{-1}(b_i) + (v_i - b_i) \frac{d}{db_i} b^{-1}(b_i) = 0$$

$$\Rightarrow b(v_i) = \frac{1}{2} v_i$$

$$-b^{-1}(b_i) + (v_i - b_i) \frac{d}{db_i} b^{-1}(b_i) = 0$$

$$\Leftrightarrow -v_i + (v_i - b(v_i)) \frac{d}{db(v_i)} = 0$$

$$\Leftrightarrow -v_i + (v_i - b(v_i)) \frac{1}{b'(v_i)} = 0$$

$$\Leftrightarrow b'(v_i) v_i + b(v_i) = v_i$$

$$\Leftrightarrow b(v_i) v_i = \frac{1}{2} v_i^2 + k; b(0) = 0$$

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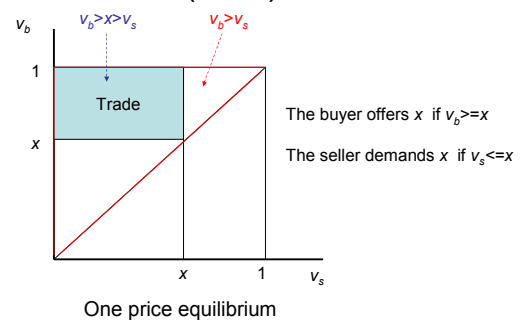
Application 2: A Double Auction

- The buyer's valuation for the seller's good is v_b , the seller's is v_s . These valuations are private information and are drawn from independent **uniform distributions on [0,1]**
- The seller names an asking price, p_s , and the buyer simultaneously names an offer price, p_b
- If $p_b \geq p_s$, then trade occurs at price $p = (p_b + p_s)/2$; if $p_b < p_s$ then no trade occurs

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Application 2: A Double Auction (cont')



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Application 2: A Double Auction (cont')

- A pair of strategies $\{p_b(v_b), p_s(v_s)\}$ is a Bayesian Nash equilibrium if the following two conditions hold

$$\max_{p_b} \left[v_b - \frac{p_b + E[p_s(v_s) | p_b \geq p_s(v_s)]}{2} \right] \text{prob}\{p_b \geq p_s(v_s)\}$$

Expected price the seller will demand

$$\max_{p_s} \left[\frac{p_s + E[p_b(v_b) | p_b(v_b) \geq p_s]}{2} - v_s \right] \text{prob}\{p_b(v_b) \geq p_s\}$$

Expected price the buyer will offer

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Application 2: A Double Auction (cont')

Assume linear BNE $p_s(v_s) = a_s + c_s v_s$; $p_b(v_b) = a_b + c_b v_b$

p_b and p_s can be derived by solving

$$\max_{p_b} \left[v_b - \frac{1}{2} \left(p_b + \frac{a_s + p_b}{2} \right) \right] \frac{p_b - a_s}{c_s}$$

$$\max_{p_s} \left[\frac{1}{2} \left(p_s + \frac{p_s + a_b + c_b}{2} \right) - v_s \right] \frac{a_b + c_b - p_s}{c_b}$$

Solve FOC simultaneously

$$p_b = \frac{2}{3} v_b + \frac{1}{3} a_s$$

$$p_s = \frac{2}{3} v_s + \frac{1}{3} (a_b + c_b)$$

$$\Rightarrow p_b = \frac{2}{3} v_b + \frac{1}{12}$$

$$p_s = \frac{2}{3} v_s + \frac{1}{4}$$

$$a_s + c_s \cdot 0 \leq p_s(v_s) \leq p_b$$

$$\Leftrightarrow E[p_s(v_s) | p_b \geq p_s(v_s)] = \frac{a_s + p_b}{2}$$

$$p_s \leq p_b(v_b) \leq a_b + c_b \cdot 1$$

$$\Leftrightarrow E[p_b(v_b) | p_b(v_b) \geq p_s] = \frac{p_s + a_b + c_b}{2}$$

$$\text{prob}\{p_b \geq p_s(v_s)\} = \text{prob}\left\{v_s \leq \frac{p_b - a_s}{c_s}\right\}$$

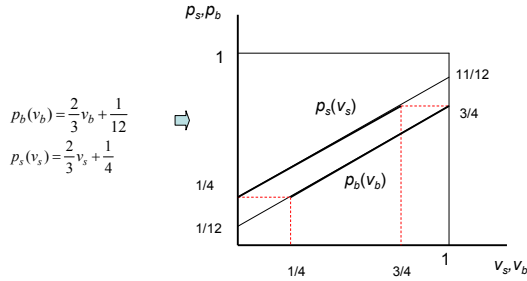
$$\text{prob}\{p_b(v_b) \geq p_s\} = 1 - \text{prob}\left\{v_b \leq \frac{p_s - a_b}{c_b}\right\}$$

$$= \frac{a_b + c_b - p_s}{c_b}$$

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Application 2: A Double Auction (cont')



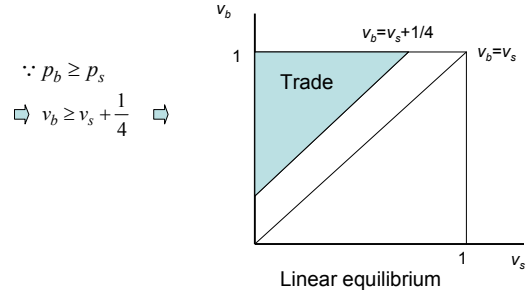
$$p_b(v_b) = \frac{2}{3}v_b + \frac{1}{12}$$

$$p_s(v_s) = \frac{2}{3}v_s + \frac{1}{4}$$

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Application 2: A Double Auction (cont')



$$\because p_b \geq p_s$$

$$\Rightarrow v_b \geq v_s + \frac{1}{4} \Rightarrow$$

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Direct Mechanisms

- Direct Mechanisms
 - Static Bayesian games in which each player's only action is to **submit a claim about his or her type**
 - A new static Bayesian game with the same types spaces and beliefs as original static Bayesian game but with **new action spaces** and **new payoff functions**
 - The payoff functions are chosen so as to confront each player with a choice of exactly of its kind

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The Revelation Principle (Cont')

- Direct Mechanisms Design
 - Consider the Bayesian Nash equilibrium $s^* = (s_1^*, \dots, s_n^*)$ in static Bayesian game
 - $G = \{A_1, \dots, A_n; T_1, \dots, T_n; p_1, \dots, p_n; u_1, \dots, u_n\}$
 - A direct mechanism is a new static Bayesian game is $G = \{T_1, \dots, T_n; T_1, \dots, T_n; p_1, \dots, p_n; v_1, \dots, v_n\}$

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The Revelation Principle (Myerson 1979)

- Incentive-compatible direct mechanisms
 - **Truth-telling** is a Bayesian Nash equilibrium
- Theorem (The revelation Principle)
 - Any Bayesian Nash equilibrium of any Bayesian game can be represented by an incentive-compatible direct mechanism

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Direct Mechanism

- Implementation of a direct mechanism
 - (1) Each player signs a contract that allows the neutral outsider to dictate the action you will take when we later play the game G
 - (2) Each of the players write down a claim about your type τ_i , and submit to the neutral outsider
 - (3) The neutral outsider use each player's type report in new game τ_i , together with the player's equilibrium strategy from the old game s_i^* , to compute the action the player would have taken in the equilibrium s^* if the player's type's type really were
 - (4) The neutral outsider will dictate that each of the players to take the action the neutral outsider have computed for the players, and the players will receive the resulting payoff

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The Revelation Principle (Cont')

- The Revelation Principle (truth-telling) is a Bayesian Nash equilibrium (Proof)
 - Player i 's action s_i and the players' type report $\tau = (t_1, \dots, t_n)$
 - Player i 's payoff $v_i(\tau, t_i) = u_i[s^*(\tau), t_i]$, where $t = (t_1, \dots, t_n)$
 - For each of i 's types t_i in T_i , $s_i^*(t_i)$ is the best action for i to choose from A_i , given that other players' strategies are $(s_1^*, \dots, s_{i-1}^*, s_{i+1}^*, \dots, s_n^*)$
 - If other players tell the truth, then when i 's type is t_i the best type to claim to be is t_i . The new Bayesian Nash equilibrium of is for each player i to play the truth-telling strategy $s_i(t_i) = t_i$ for every t_i in T_i



Homework #3

- Problem set
 - 3.2, 3.3, 3.6, 3.7, 3.8 (from Gibbons)
- Due date
 - two weeks from current class meeting
- Bonus credit
 - Propose new applications in the context of IT/IS or potential extensions from examples discussed

