Game theory

- The study of multiperson decisions
- Four types of games
  - Static games of complete information
  - Dynamic games of complete information
  - Static games of incomplete information
  - Dynamic games of incomplete information
- Static v. dynamic
  - Simultaneously v. sequentially
- Complete v incomplete information
  - Players' payoffs are public known or private information

Concept of Game Equilibrium

- Nash equilibrium (NE)
  - Static games of complete information
- Subgame perfect Nash equilibrium (SPNE)
  - Dynamic games of complete information
- Bayesian Nash equilibrium (BNE)
  - Static games of incomplete information
- Perfect Bayesian equilibrium (PBE)
  - Dynamic games of incomplete information

© 2008 Institute of Information Management National Chiao Tung University

Lecture Notes II-1 Static Games of Complete Information

- Normal form game
- The prisoner's dilemma
- Definition and derivation of Nash equilibrium
- Cournot and Bertrand models of duopoly
- Pure and mixed strategies

© 2008 Institute of Information Management National Chiao Tung University

Static Games of Complete Information

- First the players simultaneously choose actions; then the players receive payoffs that depend on the combination of actions just chosen
- The player's payoff function is common knowledge among all the players

© 2008 Institute of Information Management National Chiao Tung University

Normal- Form Representation

- The normal-form representation of a games specifies
  - (1) the players in the game
  - (2) the strategies available to each player
  - (3) the payoff received by each player for each combination of strategies that could be chosen by the players

© 2008 Institute of Information Management National Chiao Tung University

Game definition

- Denotation
  - n player game
  - Strategy space S_i : the set of strategies available to player i
  - $s_i \in S_i$: s_i is a member of the set of strategies S_i
  - Player i's payoff function $u_i(s_1,\ldots,s_n)$: the payoff to player i if players choose strategies $(s_1,\ldots,s_n)$
- Definition
  - The normal-form representation of an n-player game specifies the players' strategy spaces $S_1,\ldots,S_n$ and their playoff functions $u_1,\ldots,u_n$. We denote game by $G=\{S_1,\ldots,S_n ; u_1,\ldots,u_n\}$

© 2008 Institute of Information Management National Chiao Tung University

Example: The Prisoner's Dilemma

Prisoner 2

Prisoner 1		Mum (silent)	Fink (confess)
	Mum	-1,-1	-9,0
	Fink	0,-9	-6,-6

Strategy sets

$S_1=S_2=$**{Mum, Fink }**

Payoff functions

u_1(Mum, Mum)= -1, u_1(Mum, Fink)= -9, u_1(Fink, Mum)=0, u_1(Fink, Fink)= -6

u_2(Mum, Mum)= -1, u_2(Mum, Fink)=0, u_2(Fink, Mum)= -9, u_2(Fink, Fink)= -6

Example: The Prisoner's Dilemma

Prisoner 2

Prisoner 1		Mum (silent)	Fink (confess)
	Mum	-1,-1 (R,R)	-9,0 (S,T)
	Fink	0,-9 (T,S)	-6,-6 (P,P)

⇨ T (temptation)>R (reward)>P (punishment)>S (suckers)
(Fink, Fink) would be the outcome

Strictly Dominated Strategies

- Definition
  - In the normal-form game $G=\{S_1,\ldots,S_n ; u_1,\ldots,u_n\}$, let s_i' and s_i'' be feasible strategies for player i . Strategies s_i' is strictly dominated by strategy s_i'' if for each feasible combination of the other plays' strategies, i's payoff from playing s_i' is strictly less than i's payoff from paying s_i'' :

    $u_i(s_i,\ldots,s_{i-1}, s_i', s_{i+1},\ldots,s_n) < u_i(s_i,\ldots,s_{i-1}, s_i'', s_{i+1},\ldots,s_n)$

    for each $(s_i,\ldots,s_{i-1},s_{i+1},\ldots,s_n)$ that can be constructed from the other players' strategy spaces $S_1,\ldots,S_{i-1},\ldots,S_n$

Iterated Elimination of Dominated Strategies

	L	M	R
U	1,0	1,2	0,1
D	0,3	0,1	2,0

⇨

	L	M	R
U	1,0	1,2	0,1
D	0,3	0,1	2,0

⇩

	L	M	R
U	1,0	1,2	0,1
D	0,3	0,1	2,0

⇦

	L	M	R
U	1,0	1,2	0,1
D	0,3	0,1	2,0

Outcome =(U,M)

Weakness of Iterated Elimination

- Assume it is common knowledge that the players are rational
  - All players are rational and all players know that all players know that all players are rational.
- The process often produces a very imprecise prediction about the play of the game
- Example

	L	*C*	*R*
T	0,4	4,0	5,3
M	4,0	0,4	5,3
B	3,5	3,5	6,6

⇨ No strictly dominated strategy was eliminated

Concept of Nash Equilibrium

- Each player's predicted strategy must be that player's best response to the predicated strategies of the other players
- Strategically stable or self–enforcing
  - No single wants to deviate from his or her predicated strategy
- A unique solution to a game theoretic problem, then the solution must be a Nash equilibrium

Definition of Nash Equilibrium

- Definition
  - In the n-player normal-form game $G=\{S_1,\ldots,S_n ; u_1,\ldots,u_n\}$, the strategies $(s_1^*,\ldots,s_n^*)$ are a Nash equilibrium if, for each play i, s_i^* is player i's best response to the strategies specified for the n-1 other players, $(s_1^*,\ldots, s_{i-1}^*, s_{i+1}^*,\ldots,s_n^*)$

$$u_i(s_1^*,\ldots, s_{i-1}^*, s_i^*, s_{i+1}^*,\ldots,s_n^*) \geq u_i(s_1^*,\ldots, s_{i-1}^*, s_i, s_{i+1}^*,\ldots,s_n^*)$$

for every feasible strategy s_i in S_i ; that is s_i^* solves

$$\max_{s_i \in S_i} u_i(s_1^*,\ldots, s_{i-1}^*, s_i, s_{i+1}^*,\ldots,s_n^*)$$

Examples of Nash Equilibrium

	Mum	Fink
Mum	-1,-1	-9,0
Fink	0,-9	-6,-6

	Opera	Fight
Opera	2,1	0,0
Fight	0,0	1,2

	L	C	R
T	0,4	4,0	5,3
M	4,0	0,4	5,3
B	3,5	3,5	6,6

Examples of Nash Equilibrium

Player 2 's strategy
(best response function)

$BR_2(T)=L$
$BR_2(M)=C$
$\boldsymbol{BR_2(B)=R}$

Player 1's strategy
(best response function)

$BR_1(L)=M$
$BR_1(C)=T$
$\boldsymbol{BR_1(R)=B}$

	L	C	R
T	0,4	4,0	5,3
M	4,0	0,4	5,3
B	3,5	3,5	6,6

Application 1
Cournot Model of Duopoly

- q_1,q_2 denote the quantities (of a homogeneous product) produced by firm 1 and 2
- Demand function $P(Q)=a\text{-}Q$
  - $Q=q_1+q_2$
- Cost function $C_i(q_i)=cq_i$
- Strategy space $S_i=[0,\infty)$
- Payoff function $\pi_i(q_i,q_j)=q_i[P(q_i+q_j)-c]=q_i[a-(q_i+q_j)-c]$

Firm i 's decision

$$\max_{0\le q_i\le\infty} \pi_i(q_i,q_j^*)=\max_{0\le q_i\le\infty} q_i[a-(q_i+q_j^*)-c]$$

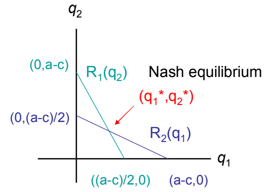
First order condition $q_i=\frac{1}{2}(a-q_j^*-c)$

$$\Rightarrow \quad q_1^*=\frac{1}{2}(a-q_2^*-c) \qquad q_2^*=\frac{1}{2}(a-q_1^*-c) \quad \Rightarrow \quad q_1^*=q_2^*=\frac{a-c}{3}$$

Cournot Model of Duopoly (cont')

- Best response functions

$$R_2(q_1)=\frac{1}{2}(a-q_1-c) \qquad R_1(q_2)=\frac{1}{2}(a-q_2-c)$$



Application 2
Bertrand Model of Duopoly

- Firm 1 and 2 choose prices p_1 and p_2 for differentiated products
- Quantity that customers demand from firm i is $q_i(p_i,p_j)=a-p_i+bp_j$
- Pay off (profit) functions

$$\pi_i(p_i,p_j)=q_i(p_i,p_j)[p_i-c]=[a-p_i+bp_j][p_i-c]$$

- Firm i's decision

$$\max_{0\le p_i\le\infty} \pi_i(p_i,p_j^*)=\max_{0\le p_i\le\infty}[a-p_i+bp_j^*][p_i-c]$$

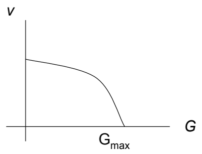
First order condition

$$p_i^*=\frac{1}{2}(a+bp_j^*+c)$$

$$\Rightarrow \quad p_1^*=\frac{1}{2}(a+bp_2^*+c), \quad p_2^*=\frac{1}{2}(a+bp_1^*+c) \quad \Rightarrow \quad p_1^*=p_2^*=\frac{a+c}{2-b}$$

Application 3
The problem of Commons

- n farmers in a village
- g_i :The number of goats owns by farmer i
- The total numbers of goats $G= g_1+\ldots+g_n$
- The value of a goat = $v(G)$
- $v'(G)<0$, $v''(G)<0$



The Problem of Commons (cont')

Firm i's decision

$$\pi_i(g_i, g_{-i}^*) = g_i v(g_i + g_{-i}^*) - cg_i$$

Fist order condition

$$\partial\pi_i(g_i, g_{-i}^*)/\partial g_i = v(g_i + g_{-i}^*) + g_i v'(g_i + g_{-i}^*) - c = 0, \forall i \in \{1,\ldots,n\}$$

Summarize all equations

$$\Rightarrow \quad v(G^*) + \frac{1}{n} G^* v'(G^*) - c = 0$$

Social optimum

$$\max_{0\le G\le\infty} Gv(G) - Gc$$

$$\Rightarrow \quad v(G^{**}) + G^{**} v'(G^{**}) - c = 0$$

$\Rightarrow$ Implication : $G^* > G^{**}$ (over grows)

Application 4
Final-Offer Arbitration

- The firm and union make wage offers w_f and w_u and then the arbitrator choose that offer that is close to his ideal settlement x
- Arbitrator knows x but the parties do not
  - The parties believe that x is a randomly distributed according to a CDF $F(x)$ and PDF $f(x)$

Application 4
Final-Offer Arbitration

- Expected wage settlement

$$prob(w_f \text{ is chosen}) = \Pr ob\left\{x < \frac{w_f + w_u}{2}\right\} = F\left(\frac{w_f + w_u}{2}\right)$$

$$prob(w_u \text{ is chosen}) = 1 - F\left(\frac{w_f + w_u}{2}\right)$$

$$\Rightarrow \quad w(w_f, w_u) = w_f \cdot F\left(\frac{w_f + w_u}{2}\right) + w_u \cdot \left[1 - F\left(\frac{w_f + w_u}{2}\right)\right]$$

- Union and firm's decisions

$$\min_{w_f} w(w_f, w_u^*) = w_f \cdot F\left(\frac{w_f + w_u^*}{2}\right) + w_u^* \cdot \left[1 - F\left(\frac{w_f + w_u^*}{2}\right)\right]$$

$$\max_{w_u} w(w_f^*, w_u) = w_f^* \cdot F\left(\frac{w_f^* + w_u}{2}\right) + w_u \cdot \left[1 - F\left(\frac{w_f^* + w_u}{2}\right)\right]$$

Final-Offer Arbitration (cont')

Solve FOC simultaneously

$$(w_u^* - w_f^*) \cdot \frac{1}{2} f\left(\frac{w_f^* + w_u^*}{2}\right) = F\left(\frac{w_f^* + w_u^*}{2}\right)$$

$$(w_u^* - w_f^*) \cdot \frac{1}{2} f\left(\frac{w_f^* + w_u^*}{2}\right) = 1 - F\left(\frac{w_f^* + w_u^*}{2}\right)$$

$$\Rightarrow \quad F\left(\frac{w_f^* + w_u^*}{2}\right) = \frac{1}{2} \qquad w_u^* - w_f^* = \frac{1}{f\left(\frac{w_f^* + w_u^*}{2}\right)}$$

The information of $w_f^* + w_u^*$ and $w_u^* - w_f^*$ with respect to CDF and PDF

Final-Offer Arbitration (cont')

Special distribution function of x

If $f(x) \sim N(m, \sigma^2)$ $\quad f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{1}{2\sigma^2}(x-m)^2\}$

$$\Rightarrow \quad \begin{aligned} w_u^* + w_f^* &= 2m \\ w_u^* - w_f^* &= \sqrt{2\pi\sigma^2} \end{aligned} \quad \Rightarrow \quad \begin{aligned} w_u^* &= m + \sqrt{\frac{\pi\sigma^2}{2}} \\ w_F^* &= m - \sqrt{\frac{\pi\sigma^2}{2}} \end{aligned}$$

$\Rightarrow$ Implication: higher uncertainty, more aggressive offer

Mixed Strategies

- Matching pennies

	Head	*Tails*
Head	-1,1	1,-1
Tails	1,-1	-1,1

- In any game in which each player would like to outguess the other(s), there is no *pure strategy* Nash equilibrium
  - E.g. poker, baseball, battle
  - The solution of such a game necessarily involves *uncertainty* about what the players will do
  - Solution : *mixed strategy*

Definition of Mixed Strategies

- Definition
  - In the normal-form game $G=\{S_1,\ldots,S_n ; u_1,\ldots,u_n\}$, suppose $S_i=\{s_{i1},\ldots,s_{iK}\}$. Then the mixed strategy for player i is a probability distribution $p_i=(p_{i1},\ldots,p_{ik})$, where $0 \le p_{ik} \le 1$ for $k=1,\ldots,K$ and $p_{i1}+\ldots+p_{iK}=1$
- Example
  - In penny matching game, a mixed strategy for player i is the probability distribution $(q,1\text{-}q)$, where q is the probability of playing Heads, $1\text{-}q$ is the probability of playing Trail, and $0 \le q \le 1$

Mixed strategy in Nash Equilibrium

- Strategy set $S_1=\{s_{11},\ldots,s_{1j}\}$, $S_2=\{s_{21},\ldots,s_{2k}\}$
- Player 1 believes that player 2 will play the strategies $(s_{21},\ldots,s_{2k})$ with probabilities $(p_{21},\ldots,p_{2k})$, then player 1's expected payoff from playing the pure strategy s_{1j} is

$$\sum_{k=1}^{k} p_{2k} u_1(s_{1j}, s_{2k})$$

- Player 1's expected payoff from paying the mixed strategy $p_1=(p_{11},\ldots,p_{1j})$ is

$$v_1(p_1,p_2)=\sum_{j=1}^{J}\sum_{k=1}^{k} p_{1j}\cdot p_{2k}u_1(s_{1j},s_{2k}) \quad v_2(p_1,p_2)=\sum_{j=1}^{J}\sum_{k=1}^{k} p_{1j}\cdot p_{2k}u_2(s_{1j},s_{2k})$$

- Definition
  - In the two player normal-form game $G=\{S_1,S_2;u_1,u_2\}$, the mixed strategies (p_1^*,p_2^*) are a Nash equilibrium if each player's mixed strategy is a best response to the other player's mixed strategy. That is

$$v_1(p_1^*,p_2^*) \ge v_1(p_1,p_2^*) \qquad v_2(p_1^*,p_2^*) \ge v_2(p_1^*,p_2)$$

Mixed Strategy

Player 2

		q *Head*	*1-q* *Tails*
Player 1 *r*	*Head*	-1,1	1,-1
1-r	*Tails*	1,-1	-1,1

Player 1's expected playoff $=q(-1)+(1-q)(1)=1-2q$ when he play *Head*
$=q(1)+(1-q)(-1)=2q-1$ when he play *Tail*

Compare $1-2q$ and $2q-1$
If $q<1/2$, then player 1 plays *Head*
If $q>1/2$, then play 1 plays *Tail*
If $q=1/2$, player 1 is indifferent in *Head* and *Tail*

Mixed strategy in Nash Equilibrium: example

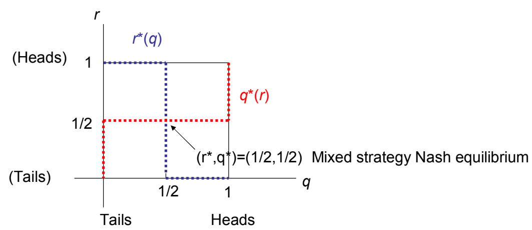
Player 2

		q *Head*	*1-q* *Tails*
Player 1 *r*	*Head*	-1,1	1,-1
1-r	*Tails*	1,-1	-1,1

Player 1's expected playoff $=rq^*(-1)+r(1-q^*)(1)+(1-r)q^*(1)+(1-r)(1-q^*)(-1)$
$=(2q^*-1)+r(2-4q^*)$
Player 2's expected playoff $=qr^*(1)+q(1-r^*)(-1)+(1-q)r^*(-1)+(1-q)(1-r^*)(1)$
$=(2r^*-1)+q(2-4r^*)$

$r^*=1$ if $q^*<1/2$ — $q^*=1$ if $r^*<1/2$
$r^*=0$ if $q^*>1/2$ — $q^*=0$ if $r^*>1/2$
$r^*=$ any number in $(0,1)$ if $q^*=1/2$ — $q^*=$ any number in $(0,1)$ if $r^*=1/2$

Mixed Strategy in Nash Equilibrium: example (cont')



Example 2

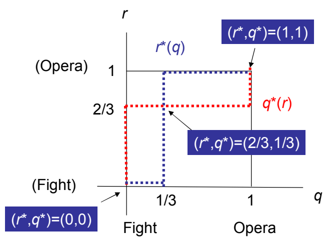
		q *Opera*	$1-q$ *Fight*
r	*Opera*	2,1	0,0
$1-r$	*Fight*	0,0	1,2

Player 1's expected playoff $=rq^*(2)+r(1-q^*)(0)+(1-r)q^*(0)+(1-r)(1-q^*)(1)$
$= r(3q^*-1)-(1+q^*)$

Player 2's expected playoff $=qr^*(1)+q(1-r^*)(0)+(1-q)r^*(0)+(1-q)(1-r^*)(2)$
$=q(3r^*-2)+1-r^*$

 National Chiao Tung University

Example 2



Payer 1's mixed strategies $(r^*,1-r^*)=(2/3,1/3)$
Payer 2's mixed strategies $(q^*,1-q^*)=(1/3,2/3)$

pure strategies $(r^*,1-r^*)=(1,0),(0,1)$
pure strategies $(q^*,1-q^*)=(1,0),(0,1)$

 National Chiao Tung University

Theorem: Existence of Nash Equilibrium

- (Nash 1950): In the n-player normal-form game $G=\{S_1,\ldots,S_n\ ;\ u_1,\ldots,u_n\}$, *if* n is finite and S_i is finite for every I then there exists at least one Nash equilibrium, possibly involving mixed strategies

 National Chiao Tung University

Homework #1

- Problem set
  - 1.3, 1.5, 1.6, 1.7, 1.8, 1.13(from Gibbons)
- Due date
  - two weeks from current class meeting
- Bonus credit
  - Propose new applications in the context of IT/IS or potential extensions from Application 1-4

 National Chiao Tung University